# PHYS CS 5: Non-Equilibrium Statistical Mechanics - Problem Set 3

Sidharth Kannan

Winter 2026

1. **Entropy Always *Decreases?!*** *(10 pts)*

   In our derivation of the $H$-theorem, we showed that $H$, the negative entropy, always decreases. This relied on the particular form of the collision integral, where we assumed that the scattering rate was proportional to the distribution function before the collisions, $f_2(\vec{r}, \vec{r}, \vec{p}_1, \vec{p}_2)$.

   Either argue or explicitly show that if we instead assume that the scattering rate is proportional the distribution *after* the collisions, $f_2(\vec{r}, \vec{r}, \vec{p}_1', \vec{p}_2')$, that $H$ increases with time and thus entropy always decreases.

2. **Streaming and Entropy** *(10 pts)*

   In the derivation of the $H-$theorem, we have the following step:

   $$\frac{dH}{dt} = \int d^3r d^3p \log f_1 \cdot \left[\{H_1, f_1\} + \left(\frac{\partial f_1}{\partial t}\right)_{coll}\right]$$

   In lecture, I asserted that the term with the Poisson bracket vanished. Show this.

   *Hint: write out the Hamiltonian, evaluate the Poisson bracket with the distribution function, and then integrate twice by parts.*

3. **A Modified Kac Ring** (taken from here) *(10 pts)*

   A Kac ring with $N$ sites is initially occupied by $B$ black and $W$ white balls. The markers are distributed at random, each edge carrying a marker with probability $\mu$. Now consider a single turn of the ring. Give an expression for the probability that the ring is occupied by only white balls at $t = 1$. How does this probability behave for large $N$?

4. **Sound Waves in the Euler Equations** *(10 pts)*

   In lecture, we didn't quite have time to go through the derivation of sound waves in an ideal fluid. Show that the Euler equations can be solved to show the existence of wave-like solutions (yes, you can just follow the lecture notes).

5. **Projection, Regression** *(20 pts)*

   Let observables be functions of phase space $f(\Gamma)$. Define an inner product

   $$(A, B) = \langle A^\dagger B \rangle,$$

   where $\langle\cdot\rangle$ denotes an equilibrium ensemble average. Let $\{A_1, \ldots, A_n\}$ be a set of resolved variables.

(a) Let $f$ be any observable. Consider approximating it by a linear combination

$$g = \sum_{i=1}^{n} \alpha_i A_i.$$

Define the mean-square error

$$E(\alpha_1, \ldots, \alpha_n) = \left\langle |f - g|^2 \right\rangle.$$

Show that minimizing $E$ with respect to $\alpha_i$ leads to the system of equations

$$\sum_{j=1}^{n} C_{ij} \alpha_j = (A_i, f),$$

where

$$C_{ij} = (A_i, A_j).$$

(b) Assume that $C$ is invertible. Solve for the coefficients $\alpha_i$ and show that the minimizing function can be written as

$$Pf = \sum_{i,j} A_i (C^{-1})_{ij} (A_j, f).$$

(c) Define the residual

$$R = f - Pf.$$

Show that

$$(A_k, R) = 0 \quad \text{for all } k.$$

Explain why this property characterizes orthogonal projection in a Hilbert space.

(d) Explain in words why $Pf$ is the best linear predictor of $f$ in the mean-square sense.