CMPTG 5 - HW 1 - Statistical Mechanics

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1. The concept of entropy was generalized to information theory by the seminal work of Claude Shannon. For some random variable x with N possible states, indexed by i, the *Shannon entropy* is defined as

$$H(x) = -\sum_i p(i) \log p(i) \tag{1}$$

   Show that the Boltzmann entropy $S = k \log \Omega$ is a special case of the Shannon entropy, in which the distribution of x is uniform ($p(i) = 1/N$.

2. Consider a simple RC circuit with no voltage source (shown below). While electrostatics tells us that without a voltage source, the charge on the capacitor plates will be 0, in practice thermal fluctuations lead to microscopic induced voltages.

   Following from our discussion of the Boltzmann Law, derive an expression for the root mean squared (RMS) voltage difference between the plates of the capacitor.

   Hint: The energy stored in a capacitor is given by $U = \frac{1}{2}C\Delta V^2$

   The resulting formula is the expression for *Johnson-Nyquist noise* (though Nyquist came to it via a much more complex argument). There are two interesting things to note.

   (a) It does not depend on the resistance of the resistor. This might lead us to think that the resistor itself is not necessary. However, this is

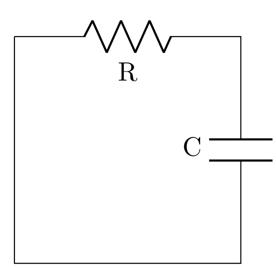

Figure 1: RC circuit with no voltage source

incorrect. The thermal fluctuations that are giving rise to our voltage all occur inside the resistor.

(b) Substitute your expression for noise voltage back into the energy equation, to get an expression for average thermal energy contained in the capacitor. The resulting expression is an example of the famous *equipartition theorem*, which states that any quadratic degree of freedom has $\frac{1}{2}kT$ units of energy associated with it.